THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 8
February 5, 2025 (Tuesday)

1 Recall

From the previous lecture notes, we discuss the projection onto convex set. Let us recall the proposition and complete the proof together.

Proposition 1. *Let K be a closed convex subset of $\mathbb{R}^n$, and $y \in \mathbb{R}^n$ be fixed, there exists a* ***unique*** *$x^* \in K$ such that*

$$\|x^* - y\| = \min_{x \in K} \|x - y\|.$$

We define $\mathrm{Proj}_K(y) := x^*$.

Proof. 1. **Existence:**
Let $f(x) := \|x - y\|^2$ for any $x \in \mathbb{R}^n$, it is clear that

$$\lim_{\|x\| \to +\infty} f(x) = +\infty$$

Then

$$\begin{cases} f(x) \text{ is coercive} \\ K \text{ is closed} \end{cases} \implies \text{there exists a minimizer for } \min_{x \in K} f(x).$$

2. **Uniqueness:**
Let x_1, x_2 with $x_1 \neq x_2$ be two minimizers of $\min_{x \in K} f(x)$, then we put

$$x_3 := \frac{x_1 + x_2}{2}$$

Now, we compute $f(x_3)$ as

$$\begin{aligned} f(x_3) &= \left\| \frac{x_1 + x_2}{2} - y \right\|^2 = \left\| \frac{x_1 - y}{2} + \frac{x_2 - y}{2} \right\|^2 \\ &= \frac{1}{4}\|x_1 - y\|^2 + \frac{1}{4}\|x_2 - y\|^2 + \frac{1}{2}\langle x_1 - y, x_2 - y \rangle \\ &< \frac{1}{4} f(x_1) + \frac{1}{4} f(x_2) + \frac{1}{2}\sqrt{f(x_1) f(x_2)} \\ &= f(x_1) \end{aligned}$$

Contradiction occurs since $x_1, x_2 \in K$ are minimizers but now we have $x_3 \in K$ is also a minimizer, so the minimizer must be unique.

□

Remarks. If $y \in K$, then $\mathrm{Proj}_K(y) := y$ which is quite trivial.

2 Example

Example 1. Consider the set $K := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, and given $y \in \mathbb{R}^n$ with $\|y\| > 1$. Find the projection of y on K.

Solution. Clearly, we guess $x^* = \dfrac{y}{\|y\|}$. To do it formally, we consider

$$f(x) = \|x - y\|^2, \;\; K := \{x \in \mathbb{R}^n : g(x) \leq 0\}, \; g(x) = \|x\|^2 - 1.$$

To check the qualification condition, we compute

$$\nabla g(x) = 2x.$$

- **Case 1:** $x = 0$,
  Then it does not satisfy the qualification.
- **Case 2:** $x \neq 0$,
  Then there exists $v \in \mathbb{R}^n$ such that
  $$\langle v, \nabla g(x) \rangle < 0$$
  So, the constraint K is qualified in this case.
  By the KKT theorem, there exists $\lambda \geq 0$ such that $\lambda g(x^*) = 0$ and
  $$\begin{aligned} \nabla f(x^*) + \lambda \nabla g(x^*) &= \mathbf{0} \\ 2(x^* - y) + \lambda \cdot 2(x^*) &= \mathbf{0} \\ (1 + \lambda)x^* &= y \\ x^* &= \frac{y}{1 + \lambda} \end{aligned}$$
  We can see that x^* and y are parallel, and if $\lambda = 0$, then $x^* = y \notin K$, which is not true.
  For $\lambda \neq 0$, then $g(x^*) = 0 \implies \|x^*\| = 1$, and this implies that
  $$x^* = \frac{y}{\|y\|}.$$

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Example 2. Consider the set $K = \{x \in \mathbb{R}^n : A^T x = b, \;\; A \in \mathbb{R}^n, \; b \in \mathbb{R}\}$, and given $y \in \mathbb{R}^n$. Find the projection of y on K. Assume $A \neq \mathbf{0}$ and $n > 1$ to exclude the trivial case.

Solution. Let $f(x) = \|x - y\|^2$, $h(x) = A^T x - b$ and $K := \{x \in \mathbb{R}^n : h(x) = 0\}$.
To check the qualification condition, we compute

$$\nabla h(x^*) = A \neq \mathbf{0} \implies \{\nabla h(x^*)\} \text{ is linearly independent}$$

There exists $v \in \mathbb{R}^n$ such that $\langle v, \nabla h(x^*) \rangle = 0$.
By the KKT Theorem, there exists $\mu \in \mathbb{R}$ such that

$$\begin{aligned} \nabla f(x^*) + \mu \nabla h(x^*) &= \mathbf{0} \\ 2(x^* - y) + \mu \cdot A &= \mathbf{0} \\ x^* &= y - \frac{\mu}{2} A \end{aligned}$$

Besides, from $x^* \in K$, so we have $A^T x^* = b$ and yields

$$\begin{aligned} A^T\left(y - \frac{\mu}{2}A\right) &= b \\ A^T y - \frac{\mu}{2}\|A\|^2 &= b \\ \frac{\mu}{2} &= \frac{A^T y - b}{\|A\|^2} \end{aligned}$$

Thus, we solve $x^* = y - \dfrac{A^T y - b}{\|A\|^2} A$. ◀

— End of Lecture 8 —